

Harmonicity of nearly cosymplectic structures

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Harmonic maps

Definition

- The energy functional of $\phi : (M, g) \rightarrow (N, h)$ is

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \nu_g.$$

- Critical points of E = harmonic maps.

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Harmonic maps

First variation

•

$$\delta E(\phi_t) = -\langle \tau(\phi), V \rangle,$$

with $V = \delta\phi_t$.

- Euler-Lagrange equation associated to E :
 - $\tau(\phi) = \text{div}_g \nabla \phi = 0$.
 - $\tau(\phi) = 0$ if and only if $\dim M = 2$.
- Characterize harmonic functions, geodesics, totally geodesic maps, biharmonic maps (between Riemannian manifolds).

Harmonic maps

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with $V = \delta\phi_t$.

- Euler-Lagrange equation associated to E :
 $\tau(\phi) = \text{trace}_g \nabla d\phi = 0$.
- Conformal invariance if $\dim M = 2$.
- Generalise harmonic functions, geodesics, totally geodesic maps, holomorphic maps (between Kähler manifolds), minimal submanifolds.

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Harmonic maps

Eells-Sampson Theorem (1964)

- If M is compact and $\text{Riem}^M \leq 0$ then there exists a harmonic representative in each homotopy class.
- Elementary proof method, importance of curvature.
- Non-existence of harmonic degree one maps from S^2 to S^1 (old method).
- Hopf map $S^3 \rightarrow S^2$.
- Holomorphic maps from S^2 to S^2 .

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- If M is compact and $\text{Riem}^M \leq 0$ then there exists a harmonic representative in each homotopy class.
- Geometric flow method, importance of curvature.
- Non-existence of harmonic degree one maps from \mathbb{T}^2 to \mathbb{S}^2 (\forall metrics).
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Harmonic maps

Vector fields

- Vector fields seen as maps from M to TM (manifold with $\dim = 2\dim M$).
- $\sigma : M \rightarrow TM$ and $d\sigma : TM \rightarrow TTM$.
- $TTM = \text{Hom } V$ where $V = \mathbb{R}^n \oplus \mathbb{R}^n$.
- $V = \text{ker } d\sigma$ for $\sigma : TM \rightarrow M$ canonical projection and $\sigma_* = \text{ker } d\sigma$ with $K = \text{span}\{\sigma_*\}$.

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Harmonic maps

Bitangent bundle

- $V(x, e) \in TM$, $V(x, e)$ and $H(x, e)$ are isomorphic to $T_x M$.
- If $(x, e) \in TM$, then vectors of $T_{(x, e)} TM$ can be written as $X + YV$ where $X, Y \in T_x M$.

• Riemann metric on TM :

$$g(X^i, Y^j) = g(X, Y)$$

$$g(X^i, V^j) = 0,$$

$$g(V^i, V^j) = g(V, V)$$

• \mathcal{G} Riemannian metric on TM , connection, curvature, etc.
 • Special relativity

Harmonic maps

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- If $(x, e) \in TM$, then vectors of $T_{(x,e)} TM$ can be written as $X^h + Y^v$ where $X, Y \in T_x M$.
- Sasaki metric on TM :

$$G(X^h, Y^h) = g(X, Y),$$

$$G(X^h, Y^v) = 0,$$

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- G Riemannian metric on TM , connection, curvature, etc...
- Sasaki rather rigid.

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Vector fields

- For $\sigma: M \rightarrow TM$,

$$d\sigma(X) = X^h + (\nabla_X \sigma)^v$$

- Energy functional for $\sigma: M \rightarrow TM$

$$E(\sigma) = \frac{1}{2} \int_M \langle d\sigma, d\sigma \rangle$$

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Vector fields

- Two variational problems for σ :
- If critical points among vector fields \rightarrow harmonic sections
 (i.e. vertical part of $\tau(\sigma) = 0$).
- If critical points among all maps from M to N \rightarrow
 harmonic maps (i.e. $\tau(\sigma) = 0$).

• Example 1:

$$\tau(\sigma) = (\nabla^* \nabla \sigma)^{\perp} + \langle \sigma, \nabla_{\sigma} \sigma \rangle^{\perp}$$

• If M is compact, harmonic section of map \rightarrow parallel.

• Example 2:

Harmonic maps

Vector fields

- Two variational problems for σ :
- i) critical points among vector fields \rightarrow harmonic sections
 \Leftrightarrow vertical part of $\tau(\sigma) = 0$.
- ii) critical points among all maps from M to $TM \rightarrow$
harmonic maps $\Leftrightarrow \tau(\sigma) = 0$.
- Tension fields

$$\tau(\sigma) = (\nabla^* \nabla \sigma)^v + (R(\sigma, \nabla_{e_i} \sigma) e_i)^h$$

- If M compact, harmonic section or map \Rightarrow parallel.
- Dead-end.

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Harmonic maps

Vector fields : alternatives

- 1) Unit sections, i.e. of T^1M .
- 2) What metrics then Sasakian, then quaternionic.
- 3) What functional problems, like maps?
- 4) What bundles \rightarrow ?

Harmonic maps

Vector fields : alternatives

- i) Unit sections, i.e. of $T^1 M$.
- ii) other metrics than Sasaki, then discussion.
- iii) other functional (volume, biharmonic).
- iv) other bundle ✓

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Homogeneous fibre bundles

Homogeneous fibre bundles (CMW)

- Let G be a Lie group, $\xi : Q \rightarrow M$ a G -principal fibre bundle.
- Let $H \subset G$ be a Lie sub-group and $N = G/H$ then $\xi : Q \rightarrow M$ is an H -principal sub-bundle.
- Then $\xi = \pi \circ \tilde{\xi}$ where $\pi : N \rightarrow M$ is fibre bundle with structure group H/H isomorphic to $G/H \cong G/H$.
- If G/H is reductive, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and \mathfrak{m} is \mathfrak{h} -invariant.
- π is equipped with a \mathfrak{m} -reduction $\tilde{\xi} : Q \rightarrow N$ and $\xi = \pi \circ \tilde{\xi}$ is a G -reduction $\xi : Q \rightarrow M$ with reductive structure group G/H .

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- Let $H \subset G$ a Lie sub-group and $N = Q/H$ then $\zeta : Q \rightarrow N$ is an H -principal sub-bundle.
- Then $\xi = \pi \circ \zeta$ where $\pi : N \rightarrow M$ is fibre bundle with fibre G/H , isomorphic to $Q \times_G G/H$.
- Assume G/H is reductive, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\text{Ad}_G(H)\mathfrak{m} \subset \mathfrak{m}$.
- M is equipped with a Riemannian metric g , G/H with G -invariant metric and Q with a \mathfrak{g} -valued connection ω .

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- Then $TN = \mathcal{V} \oplus \mathcal{H}$,
- $\mathcal{V} = \ker \alpha = \mathbb{C}(X) \oplus \mathbb{C}(Y)$ and $\mathcal{H} = \mathbb{C}(Z)$,
- Trivialization of \mathcal{V} with horizontal lift bundle isomorphism $\mathbb{C}(X) \oplus \mathbb{C}(Y) \cong \mathbb{C}(X) \oplus \mathbb{C}(Y)$,
- Canonical $d(\alpha|_E) = \nu|_E \in \text{img } \nu$ at $E \in T_x \mathcal{O}$,
- The $\mathbb{C}(Z)$ is $\mathbb{C}(N)$ if N is fixed by $\nu = \nu^*(\nu + \nu^*)$.

Homogeneous fibre bundles

Homogeneous fibre bundles

- Then $TN = \mathcal{V} \oplus \mathcal{H}$,
- $\mathcal{V} = \ker d\pi = \zeta_*(\ker \xi_*)$ and $\mathcal{H} = \zeta_*(\ker \omega_*)$.
- Trivialisation of \mathcal{V} with the canonical fibre bundle $\mathfrak{m}_Q \rightarrow N$, associated to $\zeta : Q \rightarrow N \Rightarrow I : \mathcal{V} \rightarrow \mathfrak{m}_Q$.
- Connector $\phi(\zeta_*E) = q \bullet \omega_m(E) \in \mathfrak{m}_Q$ for all $E \in T_qQ$.
- Therefore metric h on N defined by $h = \pi^*g + \langle \phi, \phi \rangle$.

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- Connector $\phi(\zeta_*E) = q \bullet \omega_m(E) \in \mathfrak{m}_Q$ for all $E \in T_qQ$.
- Therefore metric h on N defined by $h = \pi^*g + \langle \phi, \phi \rangle$.

Homogeneous fibre bundles

Homogeneous fibre bundles

- Then $TN = \mathcal{V} \oplus \mathcal{H}$,
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Homogeneous fibre bundles

Reduction

- Take a group reduction of $\xi : Q \rightarrow M$.
- Let \mathcal{H} be a principal sub-bundle $\mathcal{H} \rightarrow Q \rightarrow M$ of Q .
- The sub-bundle \mathcal{H} is G -invariant if and only if the base map $\mathcal{H} \rightarrow M$ is TG/G .
- Let $\mathcal{H} \rightarrow Q$ be a principal sub-bundle of Q with structure group H .
- Let $\mathcal{H} \rightarrow M$ be the base map.
- The sub-bundle $\mathcal{H} \rightarrow Q$ is G -invariant if and only if H is G -invariant (in the Lie algebra sense).

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Reduction

- Take a group reduction of $\xi : Q \rightarrow M$.
- i.e. an H -principal sub-bundle $\xi' : Q' \rightarrow M$ of Q .
- The submanifold $\zeta(Q') \subset N$ is transverse to the fibres of $\pi : N \rightarrow M$, i.e. $T\zeta(Q') \subset \mathcal{H}$.
- Hence $\zeta(Q')$ associates to any $x \in M$ a unique element of N , i.e. it defines a section of $\pi : N \rightarrow M$.
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Complex structures

Example : almost hermitian structures

- Q is the fibre bundle of unitary frames, $G = O(n)$ and $H = SU(k)$ ($n = 2k$).

\mathfrak{g} is the algebra of skew-symmetric matrices containing

$$\text{with } \mathfrak{h} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$$

\mathfrak{h} is the set of skew-symmetric matrices with 2×2 blocks

$$\text{with } \mathfrak{h}_i =$$

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- Q is the fibre bundle of unitary frames, $G = O(n)$ and $H = SU(k)$ ($n = 2k$).
- \mathfrak{h} is the algebra of skew-symmetric matrices commuting with $J_0 = \begin{pmatrix} \mathbb{O}_k & -\mathbb{I}_k \\ \mathbb{I}_k & \mathbb{O}_k \end{pmatrix}$
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Complex structures

The universal complex structure

- On π^*TM universal complex structure \mathcal{J} defined for $y \in N$ by
 - $\mathcal{J}(Y)$ is the submeridian of $T_{y,0}N$ whose image, in any frame of $T_y(N)$, is given by J_0 .
 - $\mathcal{J}^2 = -\text{Id}$. The bundle of skew-symmetric endomorphisms of $T(N)$.
 - The fibres $\mathcal{J}(y)$ above $y \in N$ are skew-symmetric endomorphisms of $T_y(N)$ which commute with $\mathcal{J}(y)$.
 - An element $\mathcal{J}(y)$ decomposes into
 - $$\mathcal{J}(y) = \mathcal{J}(y)_{\text{hor}} + \mathcal{J}(y)_{\text{ver}}$$

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- On $\pi^* TM$ universal complex structure \mathcal{J} defined for $y \in N$ by
- $\mathcal{J}(y)$ is the automorphism of $T_{\pi(y)}M$ whose matrix, in any frame of $\zeta^{-1}(y)$, is given by J_0 .
- \mathfrak{g}_Q is the fibre bundle of skew-symmetric endomorphisms of TM .
- The fibres $\mathfrak{h}_Q(\mathfrak{m}_Q)$ above $y \in N$ are skew-symmetric endomorphisms of $T_{\pi(y)}M$ which commute (anti-commute) with $\mathcal{J}(y)$.
- An element β of \mathfrak{g}_Q decomposes into

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Complex structures

Harmonicity of almost hermitian structures

- If σ is a section of $\pi : N \rightarrow M$ then $J = \sigma^* \mathcal{J}$ is an almost complex structure.

- Fundamental

$$\Delta(\sigma) = \frac{1}{2} \sum_{i,j} \langle \nabla_{e_i} \sigma, \nabla_{e_j} \sigma \rangle \langle \sigma, \sigma \rangle e_i \otimes e_j$$

- $\Delta(\sigma) = \frac{1}{2} \sum_{i,j} \langle \nabla_{e_i} J, \nabla_{e_j} J \rangle e_i \otimes e_j$

- So J is a harmonic section iff $\nabla^* \nabla J$ and J commute.

- J is a harmonic map if, moreover, it satisfies

$$\langle \nabla^* \nabla J, \nabla^* \nabla J \rangle = 0 \quad \forall x$$

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- Functional

$$E(\sigma) = \frac{\dim M}{2} + \frac{1}{2} \int_M \frac{1}{4} |\nabla J|^2 \nu_g.$$

- and $I(\tau^\vee \sigma) = \frac{1}{4} [\nabla^* \nabla J, J]$.
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$$g([R(E_i, Z), J], \nabla_{E_i} J) = 0, \quad \forall Z$$

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Examples harmonic structures

- If J is nearly-Kähler, i.e. $\nabla_X J(X) = 0$, then J is a harmonic map.
- If J is (1,2)-symplectic, i.e. $\nabla_X J(X, Y) = -2J(X, Y)$, then J is a harmonic section of Fuota's system.
- $\text{Ric}^J(X, Y) = \text{Ric}(X, Y) \pm Y$.

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Almost contact structures

Definition

- An almost contact structure on a Riemannian manifold (M, g) is the data :
 - A unit vector field ξ and a tensor (ϕ, η) such that
 - $\phi^2 = -\text{id} + \eta \otimes \xi$
 - $\eta(\xi) = 1$

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Almost contact structures

Examples

- Almost complex manifolds $\times S^1$.
- Hypersurfaces of an almost complex manifold M .
- $S^2 \subset \mathbb{C}P^1$ where S^2 is the unit sphere of imaginary quaternions with its vector product $u \times v = \Im(uv)$.
- $S^2 \subset \mathbb{C}P^1 \subset \mathbb{C}P^2$ (complex projective plane).
- $S^2 \subset \mathbb{R}P^2 = \mathbb{O}P^1$ equipped with $\xi = -\mathbb{R}^2 \left(\frac{\partial}{\partial x} \right)$ and $\eta = \mathbb{R}^2 \left(\frac{\partial}{\partial y} \right) + \mathbb{R} \left(\frac{\partial}{\partial z} \right)$ is an almost contact.

Almost contact structures

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- Almost complex manifolds $\times S^1$.
- Hypersurfaces of an almost complex manifold.
- $S^5 \subset S^6$ where S^6 is the unit sphere of imaginary Cayley numbers with its vector product $u \times v$.
- $J^{S^6}(X) = N \times X$ is nearly-Kähler.
- $S^5 (x^7 = 0)$ equipped with $\xi = -J^{S^6}(\frac{\partial}{\partial x^7})$ and $\phi(X) = J^{S^6}(X) + \eta(X)\frac{\partial}{\partial x^7}$, is almost contact.

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- $\mathbb{S}^5 (x^7 = 0)$ equipped with $\xi = -\mathcal{J}^{\mathbb{S}^6}(\frac{\partial}{\partial x^7})$ and $\phi(X) = \mathcal{J}^{\mathbb{S}^6}(X) + \eta(X)\frac{\partial}{\partial x^7}$, is almost contact.

Almost contact structures

Examples

- Almost complex manifolds $\times \mathbb{S}^1$.
- Hypersurfaces of an almost complex manifold.
- $\mathbb{S}^5 \subset \mathbb{S}^6$ where \mathbb{S}^6 is the unit sphere of imaginary Cayley numbers with its vector product $u \times v$.
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Almost contact structures

The angle of reduction

• Frame bundle, $G = SO(m)$

• Reduction to the group $H = U(n)$, $n = m/2$, $\dim = n - 2n + 1$

$$\text{Matrix } g = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

• $\theta = \angle(\xi, \xi_1) = \angle(\xi, \xi_2) = \angle(\xi, \xi_3) = \dots = \angle(\xi, \xi_n)$

• $\sin \theta = \cos \theta = 0$ or

$$\theta = 0 \text{ or } \theta = \frac{\pi}{2} \text{ and}$$

$$\xi = \pm \xi_1 = \pm \xi_2 = \dots = \pm \xi_n \text{ or } \xi = \pm \xi_1 = \pm \xi_2 = \dots = \pm \xi_n = 0$$

Almost contact structures

The angle of reduction

- Frame bundle, $G = SO(m)$
- Reduction to the group $H = U(n) \times 1 \Rightarrow m = 2n + 1$
- Matrix $\phi_0 = \begin{pmatrix} \mathbb{O}_n & -\mathbb{I}_n & 0 \\ \mathbb{I}_k & \mathbb{O}_k & 0 \\ 0 & \cdots & 0 \end{pmatrix}$
- $H = \{A \in G : A\phi_0A^{-1} = \phi_0\}$ and $\mathfrak{h} = \{a \in \mathfrak{g} : [a, \phi_0] = 0\}$.
- But $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$
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The angle of reduction

- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, $\text{Ad}(H)$ -invariant.
- $\mathfrak{m}_1 = -\mathfrak{g}(\mathfrak{h} \oplus \mathfrak{m}_2) + \mathfrak{h} \oplus (\mathfrak{m}_1 \oplus \mathfrak{m}_2)$.
- $\mathfrak{m}_2 = \mathfrak{g}(\mathfrak{h} \oplus \mathfrak{m}_1) - \mathfrak{h} \oplus (\mathfrak{m}_1 \oplus \mathfrak{m}_2)$, $\mathfrak{m}_2 = \mathfrak{g}(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$.
- \mathfrak{m}_1 and $\mathfrak{m}_2 \Rightarrow$ two equations for harmonic sections.
- \mathfrak{m}_1 and $\mathfrak{m}_2 \Rightarrow$ almost contact structure.
- Pull-back by $\pi: M \rightarrow H$.

Almost contact structures

The angle of reduction

- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$, $\text{Ad}(H)$ -invariant.
- $a_{\mathfrak{h}} = -\frac{1}{2}(\phi_0\{a, \phi_0\} + a \circ (\eta_0 \otimes \xi_0))$,
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Almost contact structures

Harmonic maps

- First harmonic sections equation :

$$[\bar{\nabla}^* \bar{\nabla} J, J] = 0$$

- Second harmonic sections equation :

$$\bar{\nabla}^* \bar{\nabla} J = |J|^2 J - \frac{1}{2} J \cdot \text{trace}(\bar{\nabla} J \otimes J)$$

- Harmonic maps equation :

$$[\bar{\nabla}_\alpha J, \bar{\nabla}^\alpha J, J, J] + 3 \bar{\nabla}_\alpha J \cdot \bar{\nabla}^\alpha J = 0 \quad \forall \alpha \in TM$$

Almost contact structures

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- Harmonic maps equation :

$$\langle \bar{\nabla}_{E_i} J, [\bar{R}(E_i, X), J] \rangle + 8 \langle \nabla_{E_i} \xi, R(E_i, X) \xi \rangle = 0 \quad \forall X \in TM.$$

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Energy functional

$$E(\sigma) = \frac{\dim M}{2} + \frac{1}{2} \int_M \frac{1}{4} |\bar{\nabla} J|^2 + |\nabla \xi|^2 \nu_\sigma.$$

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Almost contact structures

Examples of harmonic structures

- Hypersurface of a Kähler manifold, harmonic structure iff ξ harmonic unit vector field.
- $S^{2n+1} \subset \mathbb{C}P^{n+1}$ with Hopf vector field is harmonic structure and is harmonic.
- Sasakian manifold (Kähler when normal) has a harmonic structure, as vector and as map.

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Almost contact structures

Nearly cosymplectic structures

- Definition : $(\nabla_X \theta)(Y)$ is anti-symmetric in X and Y .
- Then \mathcal{L} is a Killing field with geodesic integral curves $(\nabla_X \theta = 0)$.
- Example: S^3 .
- A nearly cosymplectic structure then
 - a harmonic section
 - a harmonic map.

Almost contact structures

Nearly cosymplectic structures

- Definition : $(\nabla_X \theta)(Y)$ is anti-symmetric in X and Y .
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Nearly cosymplectic structures : method for section

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$$[\bar{\nabla}^* \bar{\nabla} J, J] = 0$$

and

$$\bar{\nabla}^* \bar{\nabla} \xi = |\nabla \xi|^2 \xi - \frac{1}{2} J \circ \text{trace}(\bar{\nabla} J \otimes \xi)$$

in terms of curvature

- The first harmonic section equation is equivalent to

$$\text{Ric}(\bar{X}, \bar{Y}) = \text{Ric}(\bar{X}, \bar{Y})$$

- The second harmonic section equation becomes

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Almost contact structures

Nearly cosymplectic structures : method for section

- ∇^2 : second covariant derivative

$$\theta^2 = -\text{Id} + \eta \otimes \xi$$

to obtain expressions of curvature.

$$\nabla_X \nabla_Y \xi - \nabla_X \nabla_Y \xi - \nabla_{[X, Y]} \xi =$$

$$R(X, Y)\xi + \theta^2(Y, X)\xi$$

$$\nabla_X \nabla_Y \xi - \nabla_X \nabla_Y \xi - \nabla_{[X, Y]} \xi =$$

Almost contact structures

Nearly cosymplectic structures : method for section

- θ^2 : second covariant derivative

$$\theta^2 = -\text{Id} + \eta \otimes \xi$$

to obtain expressions of curvature.

- $-R(X, Y, X, Y) + R(X, Y, \theta X, \theta Y) =$
 $|\nabla_X \theta(Y)|^2 + g^2(Y, \nabla_X \xi)$
- $R(W, X, Y, Z) - R(\theta W, \theta X, \theta Y, \theta Z) = \dots$

Almost contact structures

Nearly cosymplectic structures : method for section

- 2 : second covariant derivative

$$\theta^2 = -\text{Id} + \eta \otimes \xi$$

to obtain expressions of curvature.

- $-R(X, Y, X, Y) + R(X, Y, \theta X, \theta Y) =$
 $|\nabla_X \theta(Y)|^2 + g^2(Y, \nabla_X \xi)$
- $R(W, X, Y, Z) - R(\theta W, \theta X, \theta Y, \theta Z) = \dots$

Almost contact structures

Nearly cosymplectic structures : method for section

- 2 : second covariant derivative

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Nearly cosymplectic structures : method for section

- 3 : combining of the two :
 - The first harmonic section equation is always satisfied
 - The vector field ξ is harmonic and $\mathcal{R}(\xi, X)X = 0$
 - Hence the second harmonic section equation is also satisfied

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Nearly cosymplectic structures : maps

- Same method for

$$(\tilde{\nabla}_E J, [R(E_i, X), J]) + B(\nabla_E \xi, R(E_i, X)\xi) = 0 \quad \forall X \in TM.$$

- Equivalently that

$$\begin{aligned} & R(X, Y)Z - R(Y, X)W - R(W, Z)X \\ & - R(Z, W)Y - R(Y, Z)X - R(X, W)Z - R(X, Z)W \\ & - R(W, X)Z - R(W, Z)X - R(X, W)Z - R(X, Z)W \end{aligned}$$

Almost contact structures

Nearly cosymplectic structures : maps

- Same method for

$$\langle \bar{\nabla}_{E_i} J, [\bar{R}(E_i, X), J] \rangle + 8 \langle \nabla_{E_i} \xi, R(E_i, X)\xi \rangle = 0 \quad \forall X \in TM.$$

- Establish that

$$R(Y, X, W, Z) - R(Y, X, \theta W, \theta Z) = \\ -g((\nabla_W \theta)(Z), (\nabla_Y \theta)(X)) + g(Y, \nabla_X \xi)g(Z, \nabla_W \xi)$$

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- implies $\forall X \in \mathcal{F}$

$$\langle \tilde{\nabla}_E J, [\tilde{R}(E, X), J] \rangle = 0$$

- and in the direction of ξ

$$\langle \tilde{\nabla}_\xi J, [\tilde{R}(\xi, \xi), J] \rangle = 0$$

- finally for any vector X in TM

$$\langle \tilde{\nabla}_X J, \tilde{R}(X, X) \rangle = 0$$

- so $\tilde{\nabla} J$ is a horizontal map

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