Harmonicity of nearly cosymplectic structures

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Harmonic maps

Definition

- The energy functional of $\phi : (M, g) \to (N, h)$ is

$$E(\phi) = \frac{1}{2} \int_M |\, d\phi\,|^2 \, v_g.$$

- Critical points of E = harmonic maps

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Harmonic maps

First variation

- $$\delta E(\phi_t) = -\langle \tau(\phi), V \rangle,$$
  with $V = \delta\phi_t$.
- Euler-Lagrange equation associated to E :
  $\tau(\phi) = \mathrm{trace}_g \nabla d\phi = 0$.
- Conformal invariance if $\dim M = 2$.
- Generalise harmonic functions, geodesics, totally geodesic maps, holomorphic maps (between Kähler manifolds), minimal submanifolds.

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Eells-Sampson Theorem (1964)

- If M is compact and $\mathrm{Riem}^N \leq 0$ then there exists a harmonic representative in each homotopy class.
- Geometric flow method, importance of curvature.
- Non-existence of harmonic degree one maps from T^2 to S^2 ($\forall$ metrics).
- Hopf map S^3 in S^2.
- Holomorphic maps from S^2 in S^2.

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Vector fields

- Vector fields seen as maps from M to TM (manifold with dim = 2dim M).
- $\sigma : M \to TM$ and $d\sigma : TM \to TTM$.
- $TTM = H \oplus V$ where
- $V = \ker d\pi$ for $\pi : TM \to M$ canonical projection and
- $H = \ker K$ with $K = d(\exp \circ R_{-y} \circ \tau)$.

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Bitangent bundle

- $\forall (x, e) \in TM$, $V_{(x,e)}$ and $H_{(x,e)}$ are isomorphic to T_xM.
- If $(x, e) \in TM$, then vectors of $T_{(x,e)}TM$ can be written as $X^h + Y^v$ where $X, Y \in T_xM$.
- Sasaki metric on TM :

$$G(X^h, Y^h) = g(X, Y),$$

$$G(X^h, Y^v) = 0,$$

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- G Riemannian metric on TM, connection, curvature, etc.
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- For $\sigma : M \to TM$,

$$d\sigma(X) = X^h + (\nabla_X \sigma)^v$$

- Energy functional for $\sigma : M \to TM$

$$E(\sigma) = \frac{\dim M}{2} + \frac{1}{2}\int_M |\nabla\sigma|^2 v_g$$

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Vector fields

- Two variational problems for σ :
- i) critical points among vector fields $\to$ harmonic sections $\Leftrightarrow$ vertical part of $\tau(\sigma) = 0$.
- ii) critical points among all maps from M to TM $\to$ harmonic maps $\Leftrightarrow \tau(\sigma) = 0$.
- Tension fields

$$\tau(\sigma) = (\nabla^*\nabla\sigma)^v + (R(\sigma, \nabla_{e_i}\sigma)e_i)^h$$

- If M compact, harmonic section or map $\Rightarrow$ parallel.
- Dead end.

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Vector fields : alternatives

- i) Unit sections, i.e. of T^1M.
- ii) other metrics than Sasaki, then discussion.
- iii) other functional (volume, biharmonic).
- iv) other bundle ✓

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Homogeneous fibre bundles

Homogeneous fibre bundles (CMW)

- Let G be a Lie group, $\xi : Q \to M$ a G-principal fibre bundle.
- Let $H \subset G$ a Lie sub-group and $N = Q/H$ then $\zeta : Q \to N$ is an H-principal sub-bundle.
- Then $\xi = \pi \circ \zeta$ where $\pi : N \to M$ is fibre bundle with fibre G/H, isomorphic to $Q \times_G G/H$.
- Assume G/H is reductive, i.e. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $Ad_G(H)\mathfrak{m} \subset \mathfrak{m}$.
- M is equipped with a Riemannian metric g, G/H with G-invariant metric and Q with a $\mathfrak{g}$-valued connection ω.

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- Then $TN = \mathcal{V} \oplus \mathcal{H}$,
- $\mathcal{V} = \ker d\pi = \zeta_*(\ker \xi_*)$ and $\mathcal{H} = \zeta_*(\ker \omega_{\mathfrak{m}})$.
- Trivialisation of $\mathcal{V}$ with the canonical fibre bundle $\mathfrak{m}_Q \to N$, associated to $\zeta : Q \to N = I : \mathcal{V} \to \mathfrak{m}_Q$.
- Connector $\mathcal{K}(\zeta_* E) = g \bullet \omega_{\mathfrak{m}}(E) \in \mathfrak{m}_g$ for all $E \in T_g Q$.
- Therefore metric h on N defined by $h = \pi^* g + \langle \mathcal{K}, \mathcal{K} \rangle$.

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Homogeneous fibre bundles

- Then $TN = \mathcal{V} \oplus \mathcal{H}$,
- $\mathcal{V} = \ker d\pi = \zeta_*(\ker \xi_*)$ and $\mathcal{H} = \zeta_*(\ker \omega_*)$.
- Trivialisation of $\mathcal{V}$ with the canonical fibre bundle $\mathfrak{m}_Q \to N$, associated to $\zeta : Q \to N \Rightarrow I : \mathcal{V} \to \mathfrak{m}_Q$.
- Connector $\phi(\zeta_* E) = q \bullet \omega_{\mathfrak{m}}(E) \in \mathfrak{m}_Q$ for all $E \in T_q Q$.
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Reduction

- Take a group reduction of $\xi : Q \to M$.
- i.e. an H-principal sub-bundle $\xi' : Q' \to M$ of Q.
- The submanifold $\zeta(Q') \subset N$ is transverse to the fibres of $\pi : N \to M$, i.e. $T\zeta(Q') \subset \mathcal{H}$.
- Hence $\zeta(Q')$ associates to any $x \in M$ a unique element of N, i.e. it defines a section of $\pi : N \to M$.
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Complex structures

Example : almost hermitian structures

- Q is the fibre bundle of unitary frames, $G = O(n)$ and $H = SU(k)$ $(n = 2k)$.
- $\mathfrak{h}$ is the algebra of skew-symmetric matrices commuting with $J_n = \begin{pmatrix} 0_k & -I_k \\ I_k & 0_k \end{pmatrix}$
- $\mathfrak{m}$ is the set of skew-symmetric matrices anti-commuting with J_n

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The universal complex structure

- On $\pi^* TM$ universal complex structure $\mathcal{J}$ defined for $y \in N$ by
- $\mathcal{J}(y)$ is the automorphism of $T_{\pi(y)}M$ whose matrix, in any frame of $\zeta^{-1}(y)$, is given by J_0.
- $\mathfrak{g}_{\mathcal{O}}$ is the fibre bundle of skew-symmetric endomorphisms of TM.
- The fibres de $\mathfrak{h}_{\mathcal{O}}$ ($\mathfrak{m}_{\mathcal{O}}$) above $y \in N$ are skew-symmetric endomorphisms of $T_{\pi(y)}M$ which commute (anti-commute) with $\mathcal{J}(y)$.
- An element A of $\mathfrak{g}_{\mathcal{O}}$ decomposes into

$$\frac{1}{2}\mathcal{J}[\mathcal{J}, A] - \frac{1}{2}\mathcal{J}(A\mathcal{J})$$

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Harmonicity of almost hermitian structures

- If σ is a section of $\pi : N \to M$ then $J = \sigma^* \mathcal{J}$ is an almost complex structure.
- Functional

$$E(\sigma) = \frac{\dim M}{2} + \frac{1}{2}\int_M \frac{1}{4}|\nabla J|^2 v_g.$$

- and $\tau^v(\sigma) = \frac{1}{4}[\nabla^* \nabla J, J]$.
- So J is a harmonic section iff $\nabla^* \nabla J$ and J commute.
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Examples harmonic structures

- If J is nearly-Kähler, i.e. $\nabla_X J(X) = 0$, then J is a harmonic map.
- If J is $(1,2)$-symplectic, i.e. $\nabla J(JX, JY) = -\nabla J(X, Y)$, then J is a harmonic section iff Ricci^* is symmetric.
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Almost contact structures

Definition

- An almost contact structure on a Riemannian manifold (M, g) is the data :
- A unit vector field ξ and a tensor $(1, 1)$ φ such that :
- $$\varphi^2 = -\mathrm{Id} + \eta \otimes \xi$$

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- Hypersurfaces of an almost complex manifold.
- $\mathbb{S}^5 \subset \mathbb{S}^6$ where $\mathbb{S}^6$ is the unit sphere of imaginary Cayley numbers with its vector product $u \times v$.
- $J^{\mathbb{S}^6}(X) = N \times X$ is nearly-Kähler.
- $\mathbb{S}^5$ $(x^7 = 0)$ equipped with $\xi = -J^{\mathbb{S}^6}(\frac{\partial}{\partial x^7})$ and $\phi(X) = J^{\mathbb{S}^6}(X) + \eta(X)\frac{\partial}{\partial x^7}$, is almost contact.

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- $\mathbb{S}^5 \subset \mathbb{S}^6$ where $\mathbb{S}^6$ is the unit sphere of imaginary Cayley numbers with its vector product $u \times v$.
- $J^{\mathbb{S}^6}(X) = N \times X$ is nearly-Kähler.
- $\mathbb{S}^5$ $(x^7 = 0)$ equipped with $\xi = -J^{\mathbb{S}^6}(\frac{\partial}{\partial x^7})$ and $\phi(X) = J^{\mathbb{S}^6}(X) + \eta(X)\frac{\partial}{\partial x^7}$, is almost contact.

Almost contact structures

The angle of reduction

- Frame bundle, $G = SO(m)$
- Reduction to the group $H = U(n) \times 1$ — $m = 2n + 1$
- Matrix φ_0 : $\begin{pmatrix} 0_n & -1_n & 0 \\ 1_n & 0_n & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- $H = \{A \in G : A\varphi_0 A^{-1} = \varphi_0\}$ and $\mathfrak{h} = \{a \in \mathfrak{g} : [a, \varphi_0] = 0\}$.
- But $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$
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- $\mathfrak{m}_1$ and $\mathfrak{m}_2 \Rightarrow$ two equations for harmonic sections.
- Universal almost contact structure.
- Pull-back by $\sigma : M \to N$.

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Harmonicity equations

- First harmonic sections equation :

$$[\bar{\nabla}^*\bar{\nabla} J, J] = 0$$

- Second harmonic sections equation :

$$\nabla^*\nabla\xi = |\nabla\xi|^2 - \frac{1}{2} J \circ \mathrm{trace}(\bar{\nabla} J \otimes \xi)$$

- Harmonic maps equation :

$$\langle \bar{\nabla}_{E_i} J, [\bar{R}(E_i, X), J] \rangle + 8 \langle \nabla_{E_i}\xi, R(E_i, X)\xi \rangle = 0 \quad \forall X \in TM$$

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Examples of harmonic structures

- Hypersurface of a Kähler manifold, harmonic structure iff ξ harmonic unit vector field.
- $S^{2n+1} \subset \mathbb{C}^{n+1}$ with Hopf vector field is harmonic, as section and as map.
- Sasaki manifold (Kählerian cone) has a harmonic structure , as section and as map.

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Nearly cosymplectic structures

- Definition : $(\nabla_X \theta)(Y)$ is anti-symmetric in X and Y.
- Then ξ is a Killing field with geodesic integral curves ($\nabla_\xi \xi = 0$).
- Example S^5 in S^6.
- If nearly cosymplectic structure then
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Nearly cosymplectic structures : method for section

- 1 : re-writing of harmonicity equations

$$[\bar{\nabla}^*\bar{\nabla}J, J] = 0$$

and

$$\nabla^*\nabla\xi = |\nabla\xi|^2\xi - \frac{1}{2}J \circ \mathrm{trace}(\bar{\nabla}J \otimes \xi)$$

in terms of curvature.

- The first harmonic section equation is equivalent to :
- $\mathrm{Ricci}^*(X, Y) = \mathrm{Ricci}^*(\varphi X, \varphi Y)$
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Nearly cosymplectic structures : method for section

- 2 : second covariant derivative

$$\theta^2 = -\operatorname{Id} + \eta \otimes \xi$$

  to obtain expressions of curvature.

- $-R(X, Y, X, Y) + R(X, Y, \theta X, \theta Y) = |(\nabla_X \theta)(Y)|^2 + g^2(Y, \nabla_X \xi)$
- $R(W, X, Y, Z) - R(\theta W, \theta X, \theta Y, \theta Z) = \ldots$

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  to obtain expressions of curvature.
- $-R(X,Y,X,Y) + R(X,Y,\theta X,\theta Y) = |(\nabla_X\theta)(Y)|^2 + g^2(Y,\nabla_X\xi)$
- $R(W,X,Y,Z) - R(\theta W,\theta X,\theta Y,\theta Z) = ...$

Almost contact structures

Nearly cosymplectic structures : method for section

- 3 : combining of the two :
- The first harmonic section equation is always satisfied.
- The vector field ξ is harmonic and $R^{\mathcal{F}}(F_i, \theta F_i)\xi = 0$
- hence the second harmonic section equation is also satisfied.

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Nearly cosymplectic structures : maps

- Same method for

$$\langle \bar{\nabla}_{E_i} J, [\bar{R}(E_i, X), J] \rangle + 8 \langle \nabla_{E_i} \xi, R(E_i, X)\xi \rangle = 0 \quad \forall X \in TM.$$

- Establish that

$$R(Y, X, W, Z) - R(Y, X, \theta W, \theta Z) =$$

$$-g((\nabla_W \theta)(Z), (\nabla_Y \theta)(X)) + g(Y, \nabla_X \xi) g(Z, \nabla_W \xi)$$

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- Finally for any vector in TM

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