On the Hermitian structures of the sequence of tangent bundles of an affine manifold endowed with a Riemannian metric

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Abstract

Let $(M,\nabla,\langle\ ,\ \rangle)$ be a manifold endowed with a flat torsionless connection ∇ and a Riemannian metric $\langle\ ,\ \rangle$ and $(T^kM)_{k\geq 1}$ the sequence of tangent bundles given by $T^kM = T(T^{k-1}M)$ and $T^1M = TM$. We show that, for any $k\geq 1$, T^kM carries a Hermitian structure (J_k, g_k) and a flat torsionless connection ∇^k and when M is a Lie group and $(\nabla,\langle\ ,\ \rangle)$ are left invariant there is a Lie group structure on each T^kM such that (J_k, g_k, ∇^k) are left invariant. It is well-known that (TM, J_1, g_1) is Kähler if and only if $\langle\ ,\ \rangle$ is Hessian, i.e, in each system of affine coordinates $(x_1,\ldots,x_n)$, $\langle \partial_{x_i},\partial_{x_j}\rangle = \frac{\partial^2\phi}{\partial_{x_i}\partial_{x_j}}$. Having in mind many generalizations of the Kähler condition introduced recently, we give the conditions on $(\nabla,\langle\ ,\ \rangle)$ so that (TM, J_1, g_1) is balanced, locally conformally balanced, locally conformally Kähler, pluriclosed, Gauduchon, Vaismann or Calabi-Yau with torsion. Moreover, we can control at the level of $(\nabla,\langle\ ,\ \rangle)$ the conditions insuring that some (T^kM, J_k, g_k) or all of them satisfy a generalized Kähler condition. For instance, we show that there are some classes of $(M,\nabla,\langle\ ,\ \rangle)$ such that, for any $k\geq 1$, (T^kM, J_k, g_k) is balanced non-Kähler and Calabi-Yau with torsion. By carefully studying the geometry of $(M,\nabla,\langle\ ,\ \rangle)$, we develop a powerful machinery to build a large classes of generalized Kähler manifolds.