PROPOSAL FOR A WORKSHOP
AT LMIA
TO BE HELD IN MAY 26-29 2015

METRIC PROBLEMS IN PROJECTIVE AND GRASSMANN SPACES

1. Introduction

The workshop will be devoted to important and outstanding metric problems in projective and Grassmann spaces. It deals with equidistant points in Grassmann spaces. Several problems can be considered, however a typical problem of this kind concerns equi-isoclinic planes in Euclidean, complex or quaternionic spaces.

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$. A p-set of equi-isoclinic n-planes in $\mathbb{F}^r$ is a set of p n-planes spanning $\mathbb{F}^r$ each pair of which has the same non-zero angle $\arccos\sqrt{\lambda}$.

Lemmes and Seidel [17] pose the problem of finding the maximum number $v(n, r, \mathbb{R})$ of equi-isoclinic n-planes that can be embedded in $\mathbb{R}^r$, and more precisely, the maximum number $v_\lambda(n, r, \mathbb{R})$ of equi-isoclinic n-planes in $\mathbb{R}^r$ with the parameter λ.

In [17] the authors prove that

$$(n - r\lambda)v_\lambda(n, r, \mathbb{R}) \leq r(1 - \lambda). \tag{1}$$

They also determine the values of $v_\lambda(n, 2n, \mathbb{R})$ for all n and λ, by use of the Hurwitz matrix equations. In particular, they prove that $v(2, 4, \mathbb{R}) = 4$. Hoggar [15] proved that (1) applies to vector spaces over $\mathbb{F} = \mathbb{C}, \mathbb{H}$. He also extended the determination of $v(n, 2n, \mathbb{F})$ to $\mathbb{F} = \mathbb{C}, \mathbb{H}$. He obtained for instance $v(2, 4, \mathbb{C}) = 6$ and $v(2, 4, \mathbb{H}) = 7$. In [14], the author gave the values of $v_\lambda(2, 2r, \mathbb{R})$ for an infinite family of ordered pairs (λ, r) . In [12] he proved that $v(2, 6, \mathbb{R}) = 9$ and that if $\lambda > \frac{1}{4}$ the maximum number of equi-isoclinic planes in $\mathbb{R}^{2r}$ with parameter λ is equal to that one of equiangular lines in $\mathbb{C}^r$ with angle $\arccos\sqrt{\lambda}$; however, for odd integers r, no value of v was known yet. The first example in odd dimensional space is given in [13], where it is shown that $v(2, 5, \mathbb{R}) = v_{\frac{1}{4}}(2, 5, \mathbb{R}) = 5$. Recently, the same author proved that for any odd integer $k \geq 3$ such that $2k = p^\alpha + 1$, p odd prime, α non-negative integer, $v_{\frac{1}{2k-2}}(2, 2k - 1, \mathbb{R}) = 2k - 1$. It is seen that the plane problem in Euclidean odd dimensional spaces amounts to finding some complex symmetric square matrices of odd orders which are called complex conference matrices. A complex $n \times n$ *conference matrix* C is a matrix with $c_{ii} = 0$ and $|c_{ij}| = 1$, $i \neq j$ that satisfies

$$CC^* = (n - 1)I_n. \tag{2}$$

A second important and exciting problem is the investigation of n-tuples of equidistant points in real projective spaces with the property that the negative shape invariants [3] and the positive shape invariants of triples of the n-tuple are equal in number. Van Lint and Seidel [18] stressed the relation to discrete mathematics by proving that there is a one to one correspondence between equidistant point sets in projective space and switching classes of graphs. In [10] it is seen that the well-known self-complementary graphs represent equidistant point sets in real projective spaces with the property that the negative shape invariants and the positive shape invariants of triples of the set are equal in number. It would be interesting to prove the converse.

The methods combine linear and matrix algebra with geometry, especially distance geometry in the spirit of Blumenthal's book [2] and Seidel's selected work [19].

2. Specific goals

The problems which we would present as a proposal for the workshop are the following :

(1) Find for any $r \geq 4$, $v(2, 2r, \mathbb{R})$.
(2) Find for any $n \geq 4$, $v(n, 2n+1, \mathbb{R})$.
(3) Find $v(2, 7, \mathbb{R})$.
(4) Find the list of all regular v-tuples in $G(4, 4v)$ whose symmetry groups are isomorphic to the symmetric group S_v.
(5) Find $v(n, r, \mathbb{K})$ where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.
(6) Prove or disprove: there exists a complex conference matrix of order 7.
(7) Characterize all equidistant point sets in real projective spaces with the property that the negative shape invariants and the positive shape invariants of triples of the sets are equal in number.

The first problem is related to that of sicpovms in quantum physics theory, that is finding maximal sets of equiangular lines in $\mathbb{C}^r$. It is conjectured that their size is r^2. It would be interesting to find an analogue of the Hurwitz matrix equations to solve the second problem. The two following problems seem to be natural and manageable. The last problems are more difficult but it is important to note that all these problems are related. That means that any progress in complex or quaternionic spaces has consequences for the Euclidean spaces. It has become clear that all these geometric problems are related to a great variety of other questions of pure and applied nature. Complex matrices, beside their appearance in geometry, also show up in coding, quantum information and other theories. In [7] the author constructed an infinite family of complex symmetric conference matrices of odd orders, and in [8] he constructed an infinite family of complex Hermitian conference matrices of even orders and other complex conference matrices of even orders which are neither symmetric nor hermitian. It would be interesting to characterize them. The solution of the last problem will lead to new results in graph theory.

We mention further that the proposed study may be related to the well-known k-distance set problem [16] and [1], or at least the methods applied there can be useful for the equi-isoclinic planes question.

3. Organizers and Participants

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3.2. **Potential invited participants:**

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(3) Ulrich Brehm, University of Dresden (Germany), ulrich.brehm@tu-dresden.de

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